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Some integral theorems relating to the oscillations of bubbles

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Two integral theorems are proved which are applicable to the motion of an incompressible fluid in three dimensions. From either of these theorems one can derive the monopole component of the pressure fluctuation at infinity when a bubble undergoes non-spherical oscillations. The results confirm and generalize some recent calculations of this effect (Longuet-Higgins 1989*a*). They also provide a basis for a physical discussion of the origin of the monopole terms.

1. Introduction

In two recent papers, Longuet-Higgins (1989*a*, *b*), the author studied the far-field pressure due to the finite-amplitude distortions of an air bubble in an unbounded fluid, and showed that such oscillations can produce a monopole component of the pressure at infinity; the pressure varies as $1/r$, where r is the radial distance. In compressible fluids such as water, such pressure fluctuations may be responsible for the production of a significant level of underwater sound.

It was shown, in Longuet-Higgins (1989*a*), that the monopole component of the pressure can be related to the *spherically averaged* motion of the fluid in the neighbourhood of the bubble. This fact led the author to examine whether there exist any more general integral properties of the irrotational motion of a perfect fluid from which the pressure fluctuations at infinity can be derived.

The author has found two such integral properties which are described in §§3 and 4 below. The first relation [A, equation (3.11)] was derived quite independently by Benjamin (1987), using a Hamiltonian formulation. In the present paper, we give an alternative and more elementary proof. The method is the same as was used earlier to derive some integral theorems for flow in two dimensions (Longuet-Higgins 1983).

The second integral theorem [B, equations (4.8), (4.9) or (4.10)] also yields the pressure fluctuation p_2 at infinity, more directly than theorem A. The application of this result to asymmetric bubble oscillations is given in §5.

Both theorems A and B provide a means of extending previous calculations and results to other bubble configurations. In addition, we show in §6 that theorem B gives an insight into the physical origin of the monopole pressure terms.

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2. The method

As in Longuet-Higgins (1983) [hereafter called (I)], we use Lagrangian coordinates $\boldsymbol{x}$ attached to a fixed particle in the fluid. We have then

$$\frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t} = \nabla\Phi, \tag{2.1}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}(\nabla\Phi) = -\nabla p \tag{2.2}$$

and

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} = \tfrac{1}{2}(\nabla\Phi)^2 - p', \quad p' = p - p_{\mathrm{A}}, \tag{2.3}$$

where Φ is the velocity potential, assumed to tend to 0 at infinity, p will denote the pressure and p_{A} the ambient pressure at infinity. Note that (2.3) follows from Bernoulli's equation, together with the relation $\mathrm{D}\Phi/\mathrm{D}t = \Phi_t + (\nabla\Phi)^2$ (cf. Longuet-Higgins & Cokelet 1976).

Fourthly, we have the equation of continuity

$$\nabla^2\Phi = 0. \tag{2.4}$$

We make use of the following relation: if S is any surface moving with the fluid and enclosing a volume V, and F is any smooth function of $\boldsymbol{x}$ and t, then

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint F(\boldsymbol{x}, t)\,\mathrm{d}V = \iiint \frac{\mathrm{D}F}{\mathrm{D}t}\,\mathrm{d}V. \tag{2.5}$$

As in (I), we may define the total mass m, the centre of mass $\bar{\boldsymbol{x}}$, the momentum or impulse $\boldsymbol{I}$, the angular momentum $\boldsymbol{A}$ and the kinetic energy KE by

$$\left.\begin{aligned} m &= \iiint \mathrm{d}V, \quad m\bar{\boldsymbol{x}} = \iiint \boldsymbol{x}\,\mathrm{d}V, \\ \boldsymbol{I} &= \iiint \nabla\Phi\,\mathrm{d}V, \quad \boldsymbol{A} = \iiint \boldsymbol{x}\wedge\nabla\Phi\,\mathrm{d}V, \\ KE &= \iiint \tfrac{1}{2}(\nabla\Phi)^2\,\mathrm{d}V. \end{aligned}\right\} \tag{2.6}$$

By setting $F = 1$, $\boldsymbol{x}$, $\nabla\Phi$, $\boldsymbol{x}\wedge\nabla\Phi$ and $\tfrac{1}{2}(\nabla\Phi)^2$ in turn we derive

$$\left.\begin{aligned} &\frac{\mathrm{d}m}{\mathrm{d}t} = 0, \quad m\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} = \boldsymbol{I}, \\ &\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = -\iint p\boldsymbol{n}\,\mathrm{d}S, \quad \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = -\iint \boldsymbol{x}\wedge p\boldsymbol{n}\,\mathrm{d}S, \\ &\frac{\mathrm{d}}{\mathrm{d}t}KE = -\iint p\nabla\Phi\cdot\boldsymbol{n}\,\mathrm{d}S, \end{aligned}\right\} \tag{2.7}$$

where $\boldsymbol{n}$ denotes the unit normal to the surface S.

We shall imagine these and later results applied to the fluid contained between an interior cavity, or bubble, with surface S_{B} and an exterior surface S_{A}, which will be allowed to tend to infinity.

3. Theorem A

In (2.5) let us take $F = \nabla \cdot (\boldsymbol{x}\Phi)$. From (2.1) and (2.2) we have, since $\nabla \cdot \boldsymbol{x} = 3$,

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\nabla \cdot (\boldsymbol{x}\Phi) &= \frac{\mathrm{D}}{\mathrm{D}t}(3\Phi + \boldsymbol{x} \cdot \nabla\Phi) \\ &= 3[\tfrac{1}{2}(\nabla\Phi)^2 - p'] + [(\nabla\Phi)^2 - \boldsymbol{x} \cdot \nabla p'] \\ &= \tfrac{5}{2}(\nabla\Phi)^2 - \nabla \cdot (\boldsymbol{x}p'). \end{aligned} \tag{3.1}$$

So by (2.5)
$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint \nabla \cdot (\boldsymbol{x}\Phi)\,\mathrm{d}V = 5KE - \iiint \nabla \cdot (\boldsymbol{x}p')\,\mathrm{d}V. \tag{3.2}$$

Hence
$$\frac{\mathrm{D}}{\mathrm{D}t}\iint_{S_\mathrm{A}-S_\mathrm{B}} \Phi\boldsymbol{x} \cdot \boldsymbol{n}\,\mathrm{d}S = 5KE - \iint_{S_\mathrm{A}-S_\mathrm{B}} p'\boldsymbol{x} \cdot \boldsymbol{n}\,\mathrm{d}S. \tag{3.3}$$

But as $r \to \infty$, $\Phi = O(1/r)$, and so if we use (2.3) we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\iint \Phi\boldsymbol{x} \cdot \boldsymbol{n}\,\mathrm{d}S_\mathrm{A} + \iint p'\boldsymbol{x} \cdot \boldsymbol{n}\,\mathrm{d}S_\mathrm{A} \to 0. \tag{3.4}$$

Hence writing
$$B = -\iint_{S_\mathrm{B}} \Phi\boldsymbol{x} \cdot \boldsymbol{n}\,\mathrm{d}S \tag{3.5}$$

where the integral is over the surface of the bubble, we obtain

$$\frac{\mathrm{d}B}{\mathrm{d}t} = 5KE + \iint_{S_\mathrm{B}} p\boldsymbol{x} \cdot \boldsymbol{n}\,\mathrm{d}S. \tag{3.6}$$

The last term in (3.6) can be evaluated as follows. On the bubble surface we have

$$p = p_\mathrm{B} - T\nabla \cdot \boldsymbol{n}, \tag{3.7}$$

where p_B is the air pressure in the bubble, T is the surface tension and $\boldsymbol{n}$ is the unit normal to the bubble†. So we have

$$\iint p\boldsymbol{x} \cdot \boldsymbol{n}\,\mathrm{d}S = p_\mathrm{B}\iint \boldsymbol{x} \cdot \boldsymbol{n}\,\mathrm{d}S - T\iint (\nabla \cdot \boldsymbol{n})\,\boldsymbol{x} \cdot \boldsymbol{n}\,\mathrm{d}S. \tag{3.8}$$

Now from purely geometrical arguments (see the Appendix) we have

$$\iint \boldsymbol{x} \cdot \boldsymbol{n}\,\mathrm{d}S = 3V_\mathrm{B} \tag{3.9}$$

and
$$\iint (\nabla \cdot \boldsymbol{n})\,\boldsymbol{x} \cdot \boldsymbol{n}\,\mathrm{d}S = 2S_\mathrm{B}, \tag{3.10}$$

where V_B and S_B denote the volume and surface area of the bubble. Therefore (3.6) can also be written

$$\boxed{\frac{\mathrm{d}B}{\mathrm{d}t} = 5KE + 3p_\mathrm{B}V_\mathrm{B} - 2TS_\mathrm{B}.} \tag{3.11}$$

4. Theorem B

Unlike (3.2) which can be applied without restriction to any part of the fluid, we prove now a second theorem which requires the existence of a bubble, or at least a cavity devoid of fluid, where the origin O may be taken.

† See Lamb (1932, p. 474). Here $\boldsymbol{n}$ is considered as the unit normal to a family of surfaces $G(x, t) = 0$, of which S_A and S_B are both members.

Let us define the radial distance
$$r = |\boldsymbol{x}|. \tag{4.1}$$

Then we have
$$\frac{\mathrm{D}}{\mathrm{D}t}(\tfrac{1}{2}r^2) = \boldsymbol{x}\frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t} = \boldsymbol{x}\cdot\boldsymbol{\nabla}\Phi \tag{4.2}$$

by (2.1) and so
$$\frac{\mathrm{D}^2}{\mathrm{D}t^2}(\tfrac{1}{2}r^2) = (\boldsymbol{\nabla}\Phi)^2 - \boldsymbol{x}\cdot\boldsymbol{\nabla}p \tag{4.3}$$

by (2.2). We have also the identity
$$\frac{\mathrm{D}^2}{\mathrm{D}t^2}\left(\frac{1}{r}\right) = \frac{3}{r^3}\left(\frac{\mathrm{D}r}{\mathrm{D}t}\right)^2 - \frac{1}{r^3}\frac{\mathrm{D}^2}{\mathrm{D}t^2}(\tfrac{1}{2}r^2). \tag{4.4}$$

Since $\boldsymbol{x}\cdot\boldsymbol{\nabla}p = rp_r$ it follows from (4.3) and (4.4) that
$$\frac{1}{r}p_r = \frac{\mathrm{D}^2}{\mathrm{D}t^2}\left(\frac{1}{r}\right) + \frac{(\boldsymbol{\nabla}\Phi)^2 - 3\Phi_r^2}{r^3}. \tag{4.5}$$

Now an element of volume may be written
$$\mathrm{d}V = r^2\,\mathrm{d}r\,\mathrm{d}\Omega, \tag{4.6}$$

where $\mathrm{d}\Omega$ is an element of solid angle Ω subtended at the origin, i.e. an element of the surface of a unit sphere centred on O. So on integrating (4.5) over the volume V we find
$$\begin{aligned}\iiint p_r\,\mathrm{d}r\,\mathrm{d}\Omega &= \frac{\mathrm{D}^2}{\mathrm{D}t^2}\iiint \frac{1}{r}\,\mathrm{d}V + \iiint \frac{(\boldsymbol{\nabla}\Phi)^2 - 3\Phi_r^2}{r^3}\,\mathrm{d}V \\ &= \frac{\mathrm{D}^2}{\mathrm{D}t^2}\iiint r\,\mathrm{d}r\,\mathrm{d}\Omega + \iiint [(\boldsymbol{\nabla}\Phi)^2 - 3\Phi_r^2]\,r^{-1}\,\mathrm{d}r\,\mathrm{d}\Omega.\end{aligned} \tag{4.7}$$

Hence
$$\iint_{S_\mathrm{A}-S_\mathrm{B}} p\,\mathrm{d}\Omega = \frac{\mathrm{D}^2}{\mathrm{D}t^2}\iint_{S_\mathrm{A}-S_\mathrm{B}} \tfrac{1}{2}r^2\,\mathrm{d}\Omega + \iiint [(\boldsymbol{\nabla}\Phi)^2 - 3\Phi_r^2]\,r^{-1}\,\mathrm{d}r\,\mathrm{d}\Omega. \tag{4.8}$$

If we use a bar to denote spherical averages, this may be written
$$\boxed{[\bar{p}] = \frac{\mathrm{D}^2}{\mathrm{D}t^2}[\tfrac{1}{2}\overline{r^2}] + \int [\overline{(\boldsymbol{\nabla}\Phi)^2} - \overline{3\Phi_r^2}]\frac{\mathrm{d}r}{r}.} \tag{4.9}$$

On either of the two moving surfaces we may write $r = \alpha + \eta$, with α constant and η the radial displacement. Then the first term on the right of (4.9) becomes $(\mathrm{D}^2/\mathrm{D}t^2)(\alpha\bar{\eta} + \tfrac{1}{2}\overline{\eta^2})$. Now as $r \to \infty$ so α is of order r and η is $O(r^{-2})$, if we assume Φ is at most $O(r^{-1})$ at infinity. Hence this term tends to zero and in the limit we obtain
$$\bar{p}_\mathrm{S} - p_\mathrm{A} = \frac{\mathrm{D}^2}{\mathrm{D}t^2}(\alpha\bar{\eta} + \tfrac{1}{2}\overline{\eta^2})_\mathrm{B} - \int_{a+\eta}^{\infty} [\overline{(\boldsymbol{\nabla}\Phi)^2} - \overline{3\Phi_r^2}]\,r^{-1}\,\mathrm{d}r, \tag{4.10}$$

where a denotes the equilibrium radius of the bubble and p_S is given by
$$p_\mathrm{S} = p_\mathrm{B} - T\boldsymbol{\nabla}\cdot\boldsymbol{n} \tag{4.11}$$
as in (3.7).

5. Application to bubble oscillations

We shall now apply the foregoing results to evaluate the monopole pressure term due to the oscillations of distorted bubbles in an infinite fluid. For this purpose, either (3.11) or (4.10) may be used. Equation (4.10), however, is slightly more direct.†

We assume the pressure p_{B} in the bubble to be related to the equilibrium pressure

$$p_0 = p_{\mathrm{A}} + \frac{2T}{a} \tag{5.1}$$

by the gas law

$$p_{\mathrm{B}} = p_0\left(\frac{V_0}{V_{\mathrm{B}}}\right)^{\gamma} = p_0\left[\frac{\overline{(a+\eta)^3}}{a^3}\right]^{-\gamma}, \tag{5.2}$$

where V_{B} is the volume and γ is the ratio of the specific heats. Correct to second order, this becomes

$$p_{\mathrm{B}} = p_0(1-3\gamma\bar{h}/a), \tag{5.3}$$

where for convenience we have written

$$\bar{h} = \bar{\eta} + \overline{\eta^2}/a. \tag{5.4}$$

(For a distortion mode, $\bar{h}$ is of second order in the displacement.) The surface-tension term in (3.7) is in general given, to second order, by

$$\nabla \cdot \boldsymbol{n} = \frac{2}{a} - \frac{2\bar{h}}{a^2} + \frac{2}{a^3}\overline{\eta(2+\nabla_S^2)\,\eta}, \tag{5.5}$$

where ∇_S^2 denotes the surface Laplacian (see Longuet-Higgins 1989*a*, §4). On substituting these expressions into (4.10) and rearranging terms we obtain

$$a\left(\omega^2 + \frac{\mathrm{D}^2}{\mathrm{D}t^2}\right)\bar{h} = \frac{\mathrm{D}}{\mathrm{D}t^2}(\tfrac{1}{2}\overline{\eta^2}) + \int_a^\infty [\overline{(\nabla\Phi)^2} - 3\overline{\Phi_r^2}]\frac{\mathrm{d}r}{r} - \frac{2T}{a^3}\overline{\eta(2+\nabla_S^2)\,\eta}, \tag{5.6}$$

where

$$\omega^2 = \frac{3\gamma p_0}{a^2} - \frac{2T}{a^3}. \tag{5.7}$$

Note that ω is the radian frequency of the radial mode of oscillation of the bubble.

Now the right-hand side of (5.6) is bilinear in the displacement η and velocity potential Φ, and may be evaluated immediately to second order, knowing only the first-order (linear) expressions for η and Φ. Moreover, since $\bar{h}$ is proportional to the volume change in the bubble, it follows by conservation of mass that the displacement at infinity is equal to $(a/r)^2\bar{h}$. Hence, the pressure fluctuation at infinity is given by

$$p_2 = \frac{a^2}{r}\frac{\mathrm{D}^2\bar{h}}{\mathrm{D}t^2}. \tag{5.8}$$

For example, if we take the normal-mode oscillation given, to first order, by

$$\left.\begin{aligned} \eta &= A_n S_n(\theta,\phi)\cos\sigma_n t, \\ \Phi &= \frac{a\sigma_n}{n+1}A_n\left(\frac{a}{r}\right)^{n+1} S_n(\theta,\phi)\sin\sigma_n t, \end{aligned}\right\} \tag{5.9}$$

with

$$\sigma_n^2 = (n-1)(n+1)(n+2)\frac{T}{a^3} \tag{5.10}$$

† *Note added in proof.* Benjamin (1989) has independently used (3.11).

(see Lamb 1932, §275) and substitute into the right-hand side of (5.6) we obtain immediately a function of the time t, proportional to A_n^2. Since any spherical harmonic S_n satisfies

$$\nabla_S^2 S_n = -n(n+1)\,S_n \tag{5.11}$$

equation (5.6) yields

$$-a\left(\omega^2+\frac{D^2}{Dt^2}\right) = M\cos 2\sigma_n t + N, \tag{5.12}$$

where

$$\left.\begin{aligned} M &= \tfrac{1}{4}(n-1)(n+2)(4n-1)\,\overline{S_n^2}\,A_n^2\frac{T}{a^3},\\ N &= -\tfrac{3}{4}(n-1)(n+2)\,\overline{S_n^2}\,A_n^2\frac{T}{a^3}. \end{aligned}\right\} \tag{5.13}$$

From (5.12) it follows that

$$\bar{h} = \frac{M}{a(4\sigma_n^2-\omega^2)}\cos 2\sigma_n t - \frac{N}{a\omega^2} \tag{5.14}$$

and so, from (5.8), the pressure fluctuation at infinity is

$$p_2 = \frac{-4\sigma_n^2}{4\sigma_n^2-\omega^2}M\frac{a}{r}\cos 2\sigma_n t. \tag{5.15}$$

For the axisymmetric modes, for example, we have

$$S_n(\theta,\phi) = P_n(\cos\theta) \tag{5.16}$$

and

$$\overline{S_n^2} = \frac{1}{2n+1}. \tag{5.17}$$

It will be seen that (5.14) and (5.15) agree precisely with (6.26) and (7.1) of Longuet-Higgins (1989a).

6. The limit of plane waves

The unattenuated pressure term for surface waves in deep water can also be derived from (4.9). For we have in general

$$\frac{D^2}{Dt^2}[\tfrac{1}{3}\overline{r^3}] = \frac{1}{4\pi}\frac{D^2}{Dt^2}\iiint r^2\,dr\,d\Omega = 0 \tag{6.1}$$

by conservation of mass in the fluid. Writing $r = \alpha+\eta$ as before, we obtain

$$\frac{D^2}{Dt^2}[\tfrac{1}{2}\overline{r^2}] = \frac{D^2}{Dt^2}[\tfrac{1}{2}\overline{r^2}-\tfrac{1}{3}\overline{r^3}] = \frac{D^2}{Dt^2}[-\tfrac{1}{2}\overline{\eta^2}-\tfrac{1}{3}\overline{\eta^3}/a]. \tag{6.2}$$

On allowing the radius α to go to infinity, (4.9) reduces to

$$p_2+p_A-p_S = \frac{D^2}{Dt^2}(\tfrac{1}{2}\overline{\eta^2}). \tag{6.3}$$

This is the same expression for the unattenuated pressure in surface gravity waves as was given by Longuet-Higgins & Ursell (1948), apart from a constant static term which is negligible in the case of small bubbles. Equation (6.3) shows that the unattenuated pressure p_2 is related to changes in the potential energy of the surface waves, or equivalently to changes in the level of the centre of mass of the fluid as a whole.

This result amplifies a comment as to the physical cause of the monopole pressure term in bubble oscillations (Longuet-Higgins 1989*a*, §7). It is not the change in volume of the bubble which is the dynamical cause of the monopole pressures. Rather, the second-order pressure arising from the term $\mathrm{D}^2/\mathrm{D}t^2(\frac{1}{2}\overline{\eta^2})$, together with the surface-tension terms, causes radial displacements at infinity which then produce volume changes in the bubble.

A simple analogy may make the point clearer. Imagine a rigid, rectangular box of mass M which is suspended vertically from some fixed point by a very long spring. Inside the box is a small mass m suspended by a shorter spring from the upper face of the box. The mass m is displaced from its equilibrium position and the whole system oscillates vertically. Now one *could* say that the lower face of the box oscillates *because* it is attached to the upper face by the rigid vertical walls of the box. This would be a kinematic description. On the other hand, it makes more sense to say that the whole box is in oscillation because of the oscillation of the mass m relative to the upper face of the box. This would be a more dynamical description.

In the above analogy, the upper face of the box corresponds to the bubble surface, the lower face corresponds to the fluid at infinity, and the masses M and m correspond to the intervening fluid.

Note that if the box were laid on a rigid flat table, there would still be an oscillating force between the box and the table, even if there were no motion of the box. Similarly, if the fluid surrounding the bubble were encased in a large rigid sphere, allowing no radial movement at this distance, there would still be an oscillating pressure on the outer sphere, without any change in the volume of air within the bubble.

Note added in proof. In a very recent paper, Benjamin (1989) has pointed out that if the internal cavity were filled not with air but with an incompressible liquid then there would be no monopole component of the pressure at infinity. But this does not mean that the pressure fluctuations 'arose from' the changes in volume of the bubble. Rather, inserting the liquid places an additional constraint on the motion of the external fluid. In the above analogy it corresponds to clamping the upper face of the rectangular box, so that there is no vertical displacement of the box as a whole.

Appendix. Proof of equations (3.7) and (3.8)

Equation (3.7) follows immediately from the divergence theorem:

$$\iint \boldsymbol{x}\cdot\boldsymbol{n}\,\mathrm{d}S = \iiint \boldsymbol{\nabla}\cdot\boldsymbol{x}\,\mathrm{d}V = \iiint 3\,\mathrm{d}V. \tag{A 1}$$

To prove equation (3.8) note that

$$2\boldsymbol{n}\cdot[(\boldsymbol{x}\cdot\boldsymbol{\nabla})\boldsymbol{n}] = (\boldsymbol{x}\cdot\boldsymbol{\nabla})(\boldsymbol{n}\cdot\boldsymbol{n}) = 0 \tag{A 2}$$

since $\boldsymbol{n}$ is a unit vector. Therefore

$$\iiint \boldsymbol{\nabla}\cdot[(\boldsymbol{x}\cdot\boldsymbol{\nabla})\boldsymbol{n}]\,\mathrm{d}V = \iint \boldsymbol{n}\cdot[(\boldsymbol{x}\cdot\boldsymbol{\nabla})\boldsymbol{n}]\,\mathrm{d}S = 0, \tag{A 3}$$

which is to say

$$\iiint [\boldsymbol{\nabla}\cdot\boldsymbol{n} + (\boldsymbol{x}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\cdot\boldsymbol{n})]\,\mathrm{d}V = 0. \tag{A 4}$$

But

$$\iiint \boldsymbol{\nabla}\cdot(\boldsymbol{x}\boldsymbol{\nabla}\cdot\boldsymbol{n})\,\mathrm{d}V = \iiint [3\boldsymbol{\nabla}\cdot\boldsymbol{n} + (\boldsymbol{x}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\cdot\boldsymbol{n})]\,\mathrm{d}V. \tag{A 5}$$

On subtracting (A 4) from (A 5) we obtain

$$\iiint \nabla \cdot (x\nabla \cdot \boldsymbol{n})\,\mathrm{d}V = \iiint 2\nabla \cdot \boldsymbol{n}\,\mathrm{d}S = \iint 2\boldsymbol{n} \cdot \boldsymbol{n}\,\mathrm{d}S \tag{A 6}$$

by the divergence theorem. Since $\boldsymbol{n} \cdot \boldsymbol{n} = 1$ the result follows.

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